PART - A

Answer ANY TEN questions:

1) **Show that the relation R on the set {1, 2, 3} given by $R = \{(1,1),(2,2),(3,3),(1,2),(2,3)\}$ is not transitive.**

Ans: $(1,2),(2,3) \in R$ but $(1,3) \notin R$.

Hence R is not transitive.

2) **Let * be the binary operation on N given by $a * b =$ L. C. M of a and b. Find $5 * 7$.**

Ans: $5 * 7 = L.C.M\ of\ 5\ and\ 7 = 35.$

3) **Write the principal value branch of $cot^{-1}x$.**

Ans: $(0, \pi)$.

4) **Find the value of $\cos(sec^{-1}x + cosec^{-1}x), |x| \geq 1$.**

Ans: $cos\ (sec^{-1}x + cosec^{-1}x) = cos\frac{\pi}{2} = 0.$

5) **Define a diagonal matrix.**

Ans: A square matrix in which all the non-diagonal elements are zero is called as a diagonal matrix.

6) **Find the value of x if $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$.**

Ans: $10 - 12 = 5x - 6x$

$\Rightarrow\ -2 = -x$

$\Rightarrow x = 2.$

7) **If $y = \sin(ax + b)$, find $\frac{dy}{dx}$.**

Ans: $\frac{dy}{dx} = a.cos(ax + b).$

8) **Differentiate $y = e^{x^3}$ with respect to x.**

Ans: $\frac{dy}{dx} = (3x^2).\left(e^{x^3}\right)$

9) **Find $\int \sec x\,(\sec x + \tan x)dx$.**

Ans: $\int sec\ x(sec\ x + tan\ x)dx = \int sec^2x\ dx + \int sec\ x.tan\ x\ dx$

$= tanx + secx + C$

10) **Evaluate $\int_2^3 x^2\, dx$.**

Ans: $\int_2^3 x^2\, dx = \left[\frac{x^3}{3}\right]_2^3 = \frac{27-8}{3} = \frac{19}{3}$

11) **Find the unit vector in the direction of the vector $\vec{a} = 2\hat{\imath} + 3\hat{\jmath} + \hat{k}$.**

Ans: $\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{2\hat{\imath}+3\hat{\jmath}+\hat{k}}{\sqrt{14}}$

12) **Define collinear vectors.**

Ans: Two or more vectors are said to be collinear if they are parallel to the same line.

13) **Write the direction cosines of y – axis.**

Ans: < 0, 1, 0 >

14) **Define feasible region in a linear programming problem.**

Ans: The common region determined by all the constraints including the non-negative constraints of an LPP is called as feasible region.

15) **If $P(A) = 0.6, P(B) = 0.3$ and $P(A \cap B) = 0.2$, find $P(A|B)$.**

Ans: $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{0.2}{0.3} = \frac{2}{3}$.

PART – B

Answer ANY TEN questions:

16) **Verify whether the operation * defined on the set of rationals Q by $a * b = \frac{ab}{2}$ is associative or not.**

Ans: $(a * b) * c = \left(\frac{ab}{2}\right) * c = \frac{abc}{4}$

$a * (b * c) = a * \left(\frac{bc}{2}\right) = \frac{abc}{4}$

$\Longrightarrow (a * b) * c = a * (b * c)\ \forall a, b, c \in \mathrm{Q}$

Hence * is associative.

17) **Show that $sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2sin^{-1}x, \frac{-1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$.**

Ans: Let $x = \sin\theta \Longrightarrow \theta = sin^{-1}x$

$sin^{-1}\left(2x\sqrt{1-x^2}\right) = sin^{-1}(2\sin\theta . \cos\theta)$

$= sin^{-1}(\sin 2\theta)$

$= 2\theta = 2sin^{-1}x.$

18) **Find the value of $tan^{-1}(\sqrt{3}) - cot^{-1}(-\sqrt{3})$.**

Ans: $tan^{-1}(\sqrt{3}) - cot^{-1}(-\sqrt{3})$

$= \frac{\pi}{3} - \left(\pi - \frac{\pi}{6}\right) = \frac{\pi}{3} - \frac{5\pi}{6} = -\frac{\pi}{2}$

19) **If $X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, find X and Y.**

Ans: $X + Y + X - Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$\Longrightarrow 2X = \begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix}$

$\Longrightarrow X = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$ and $Y = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

20) **Find the area of the triangle whose vertices are $(2,7),(1,1)$, and $(10,8)$ using determinants.**

Ans: Area of triangle $= \frac{1}{2}\begin{vmatrix} 2 & 7 & 1 \\ 1 & 1 & 1 \\ 10 & 8 & 1 \end{vmatrix} = \frac{1}{2}[(-14)-(-63)+(-2)] = \frac{47}{2}$ sq.units.

21) **If $2x+3y=\sin x$, find $\frac{dy}{dx}$.**

Ans: $2x+3y=\sin x$

Differentiating w.r.t x

$\Longrightarrow 2+3\frac{dy}{dx} = cos\ x$

$\Longrightarrow \frac{dy}{dx} = \frac{cos\ x-2}{3}$.

22) **Differentiate $x^{\sin x}, x>0$ with respect to x.**

Ans: $y = x^{\sin x}$

Taking logarithm on both sides

$\Longrightarrow log\ y = (sin\ x)(log\ x)$

Differentiating w.r.t x

$\Longrightarrow \frac{1}{y}\frac{dy}{dx} = \frac{sinx}{x} + (cos\ x)(log\ x)$

$\Longrightarrow \frac{dy}{dx} = \left(x^{\sin x}\right)\left(\frac{sinx}{x} + (cos\ x)(log\ x)\right)$.

23) **Find $\frac{dy}{dx}$ if $y=\log_7(\log x)$.**

Ans: $y = \log_7(\log x) = \frac{\log_e(logx)}{\log_e 7}$

Differentiating w.r.t x

$\Longrightarrow \frac{dy}{dx} = \left(\frac{1}{log\ 7}\right)\left(\frac{1}{log\ x}\right)\left(\frac{1}{x}\right)$.

24) **Find the approximate value of $\sqrt{25.3}$.**

Ans: Let $y=\sqrt{x}$

$x=25, \Delta x = 0.3$

Then $\Delta y = \sqrt{x+\Delta x} - \sqrt{x} = \sqrt{25.3}-\sqrt{25} = \sqrt{25.3}-5$

$\Longrightarrow \sqrt{25.3} = 5+\Delta y$

Now, $\Delta y = \left(\frac{dy}{dx}\right)(\Delta x) = \frac{1}{2\sqrt{x}}(0.3) = \frac{0.3}{10} = 0.03$

$\Longrightarrow \sqrt{25.3} = 5+0.03 = 5.03$.

25) **Evaluate $\int x^2 . log\ x\ dx$.**

Ans: $\int x^2 . log\ x\ dx = (log\ x)\left(\frac{x^3}{3}\right) - \int\left(\frac{x^3}{3}\right)\left(\frac{1}{x}\right)dx$

$= \frac{x^3 . \log x}{3} - \frac{1}{3}\int x^2\ dx$

$= \frac{x^3 . \log x}{3} - \frac{x^3}{9} + C$

26) **Find $\int \frac{sin^2 x}{1+cos\, x}\, dx$**

Ans: $\int \frac{sin^2 x}{1+cos\, x}\, dx = \int \frac{(1-cos^2 x)}{1+cos\, x}\, dx = \int \frac{(1-cos\, x)(1+cos\, x)}{1+cos\, x} dx$

$= \int (1 - cos\, x) dx$

$= \int 1\, dx - \int cos\, x\, dx$

$= x - sin\, x + C$

27) **Evaluate $\int_0^{\pi/4} (\sin 2x)\, dx$**

Ans: $\int_0^{\pi/4} (\sin 2x)\, dx = -\frac{1}{2}[\cos 2x]_0^{\frac{\pi}{4}} = -\frac{1}{2}\left(\cos\frac{\pi}{2} - \cos 0\right) = \frac{1}{2}$.

28) **Find the order and degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^2 + \sin\left(\frac{dy}{dx}\right) + 1 = 0.$**

Ans: Order = 2

Degree is not defined**.**

29) **Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $\hat{\imath} + 2\hat{\jmath} - \hat{k}$ and $-\hat{\imath} + \hat{\jmath} + \hat{k}$ respectively in the ratio $2:1$ internally.**

Ans: $\vec{a} = \hat{\imath} + 2\hat{\jmath} - \hat{k}, \vec{b} = -\hat{\imath} + \hat{\jmath} + \hat{k}$

$m: n = 2: 1$

Required Vector $\vec{r} = \frac{m\vec{b}+n\vec{a}}{m+n} = \frac{2(-\hat{\imath}+\hat{\jmath}+\hat{k})+1(\hat{\imath}+2\hat{\jmath}-\hat{k})}{2+1} = -\frac{1}{3}\hat{\imath} + \frac{4}{3}\hat{\jmath} + \frac{1}{3}\hat{k}.$

30) **Find the area of the parallelogram whose adjacent sides are given $\vec{a} = 3\hat{\imath} + \hat{\jmath} + 4\hat{k}$ and $\vec{b} = \hat{\imath} - \hat{\jmath} + \hat{k}$.**

Ans: $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 1 & 4 \\ 1 & -1 & 1 \end{vmatrix} = 5\hat{\imath} + \hat{\jmath} - 4\hat{k}$

Area of the parallelogram $= |\vec{a} \times \vec{b}| = \sqrt{25 + 1 + 16} = \sqrt{42}$ sq. units

31) **Find the distance of a point $(3, -2, 1)$ from the plane $2x - y + 2z + 3 = 0$.**

Ans: $(x_1, y_1, z_1) = (3, -2, 1)$

$A = 2, B = -1, C = 2, D = -3$

Required Distance $d = \frac{|Ax_1+By_1+Cz_1-D|}{\sqrt{A^2+B^2+C^2}} = \frac{|(6)+(2)+(2)+3|}{\sqrt{4+1+4}} = \frac{13}{3}$ units.

32) **Find the angle between the pair of lines given by $\vec{r} = (3\hat{\imath} + 2\hat{\jmath} - 4\hat{k}) + \lambda(\hat{\imath} + 2\hat{\jmath} + 2\hat{k})$ and $\vec{r} = (5\hat{\imath} - 2\hat{\jmath}) + \mu(3\hat{\imath} + 2\hat{\jmath} + 6\hat{k})$.**

Ans: $\vec{b_1} = \hat{\imath} + 2\hat{\jmath} + 2\hat{k}, \vec{b_2} = 3\hat{\imath} + 2\hat{\jmath} + 6\hat{k}$

$|\vec{b_1}| = \sqrt{1 + 4 + 4} = 3, |\vec{b_2}| = \sqrt{9 + 4 + 36} = 7$

Angle between lines $\theta = cos^{-1}\left|\frac{\vec{b_1}.\vec{b_2}}{|\vec{b_1}||\vec{b_2}|}\right| = cos^{-1}\left|\frac{3+4+12}{(3)(7)}\right| = cos^{-1}\left(\frac{19}{21}\right).$

33) **The random variable X has a probability distribution P(X) of the following form, where k is some number:**

$$P(X) = \begin{cases} k, if\ X = 0 \\ 2k, if\ X = 1 \\ 3k, if\ X = 2 \\ 0, otherwise \end{cases}$$ **Determine the value of k.**

Ans:

X	0	1	2
P(X)	k	2k	3k

$k + 2k + 3k = 1$

$\Rightarrow 6k = 1$

$\therefore k = \frac{1}{6}$

PART – C

Answer ANY TEN questions

34) **Show that the relation R in the set of integers given by $R\{(a,b): 2 \text{ divides } (a-b)\}$ is an equivalence relation.**

Solution: 2 divides 0

$\Rightarrow$ 2 divides a-a

$\Rightarrow$ $(a,a) \in R \quad \forall a \in Z$ $\quad \therefore$ R is reflexive.

Let $(a,b) \in R \Rightarrow$ 2 divides a-b

$\Rightarrow$ 2 divides $-(a-b)$

$\Rightarrow$ 2 divides b-a

$\Rightarrow$ $(b,a) \in R$ $\quad \therefore$ R is symmetric.

Let $(a,b) \in R$ and $(b,c) \in R$

$\Rightarrow$ 2 divides a-b and 2 divides b-c

$\Rightarrow$ 2 divides (a-b) + (b-c)

$\Rightarrow$ 2 divids $a - c \Rightarrow (a,c) \in R$

$\therefore$ R is transitive

Thus R is Equivalence relation.

35) **Solve $\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$, $x > 0$**

Solution: $\tan^{-1}\left(\frac{2x+3x}{1-2x\times 3x}\right) = \frac{\pi}{4}$

$\tan^{-1}\left(\frac{5x}{1-6x^2}\right) = \frac{\pi}{4}$

$\frac{5x}{1-6x^2} = \tan\frac{\pi}{4}$

$\frac{5x}{1-6x^2} = 1 \Rightarrow 5x = 1 - 6x^2$

$\Rightarrow 6x^2 + 5x - 1 = 0$

$$6x^2+6x-x-1=0$$
$$6x(x+1)-1(x+1)=0$$
$$6x-1=0 \text{ or } x+1=0$$
$$x=\frac{1}{6} \text{ and } x=-1$$
since $x=-1$ does not satisfy the equation

$$\therefore\ x=\frac{1}{6} \text{ is a solution.}$$

36) Express $A=\begin{bmatrix}1 & 5\\ -1 & 2\end{bmatrix}$, as the sum of symmetric and skew symmetric matrix.

Solution: Let $A=\begin{bmatrix}1 & 5\\ -1 & 2\end{bmatrix}$, $A'=\begin{bmatrix}1 & -1\\ 5 & 2\end{bmatrix}$

$$A+A'=\begin{bmatrix}1 & 5\\ -1 & 2\end{bmatrix}+\begin{bmatrix}1 & -1\\ 5 & 2\end{bmatrix}=\begin{bmatrix}2 & 4\\ 4 & 4\end{bmatrix}$$

Let $P=\frac{1}{2}(A+A')=\frac{1}{2}\begin{bmatrix}2 & 4\\ 4 & 4\end{bmatrix}=\begin{bmatrix}1 & 2\\ 2 & 2\end{bmatrix}$

Now $p'=\begin{bmatrix}1 & 2\\ 2 & 2\end{bmatrix}=P$

Thus $P=\frac{1}{2}(A+A')$ is a symmetric matrix

$$(A-A')=\begin{bmatrix}1 & 5\\ -1 & 2\end{bmatrix}-\begin{bmatrix}1 & -1\\ 5 & 2\end{bmatrix}=\begin{bmatrix}0 & 6\\ -6 & 0\end{bmatrix}$$

Let $Q=\frac{1}{2}(A-A')=\frac{1}{2}\begin{bmatrix}0 & 6\\ -6 & 0\end{bmatrix}=\begin{bmatrix}0 & 3\\ -3 & 0\end{bmatrix}$

$$Q'=\begin{bmatrix}0 & -3\\ 3 & 0\end{bmatrix}=-Q$$

$\therefore\ Q=\frac{1}{2}(A-A')$ is skew symmetric matrix.

Now $P+Q=\begin{bmatrix}1 & 2\\ 2 & 2\end{bmatrix}+\begin{bmatrix}0 & 3\\ -3 & 0\end{bmatrix}=A$

Thus A is represented as sum of symmetric and skew symmetric matrix.

37) Without expanding and using property of determinants, prove that: $\begin{vmatrix}2 & 7 & 65\\ 3 & 8 & 75\\ 5 & 9 & 86\end{vmatrix}=0$

Solution: $\Delta=\begin{vmatrix}2 & 7 & 65\\ 3 & 8 & 75\\ 5 & 9 & 86\end{vmatrix}$

$$=\begin{vmatrix}2 & 7 & 63+2\\ 3 & 8 & 72+3\\ 5 & 9 & 81+5\end{vmatrix}$$

$$=\begin{vmatrix}2 & 7 & 63\\ 3 & 8 & 72\\ 5 & 9 & 81\end{vmatrix}+\begin{vmatrix}2 & 7 & 2\\ 3 & 8 & 3\\ 5 & 9 & 5\end{vmatrix}$$

$$=\begin{vmatrix}2 & 7 & 9(7)\\ 3 & 8 & 9(8)\\ 5 & 9 & 9(9)\end{vmatrix}+0 \qquad [\because \text{ Two columns are identical}]$$

$$=9\begin{vmatrix}2 & 7 & 7\\ 3 & 8 & 8\\ 5 & 9 & 9\end{vmatrix}$$

$= 0$ $\qquad$ [$\because$ Two columns are identical]

38) Find dy/dx, if $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ $0 < x < 1$

Solution: $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Put $x = \tan\theta \quad \Rightarrow \theta = \tan^{-1}x$

$$y = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$y = \cos^{-1}(\cos 2\theta) = 2\theta$$

$$y = 2\tan^{-1}x.$$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{2}{1+x^2}$$

39) If $x = a(\theta - \sin\theta)$ and $y = a(1+\cos\theta)$, find $\frac{dy}{dx}$

Solution: $x = a(\theta - \sin\theta)$ $\qquad$ $y = a(1+\cos\theta)$

D.w.r. to θ	D.w.r to θ
$\frac{dx}{d\theta} = a(1-\cos\theta)$	$\frac{dy}{d\theta} = a(-\sin\theta)$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-a\sin\theta}{a(1-\cos\theta)}$$

$$\frac{dy}{dx} = \frac{-2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = -\cot\frac{\theta}{2}$$

40) Verify Mean value theorem for the function $f(x)=x^2$ in the interval [2,4].

Solution: The function $f(x)=x^2$ is continuous in [2,4]

and differentiable in (2,4) as its derivative $f^1(x)=2x$ is defined in (2,4).

Now $f(2)=4$, and $f(4)=16$

Hence there exist at least one value $c\in(2,4)$

such that $f^1(c)=\frac{f(b)-f(a)}{b-a}$

Now $f(x)=x^2 \Rightarrow f^1(x)=2x \Rightarrow f^1(c)=2c$

$$\Rightarrow \quad 2c=\frac{16-4}{4-2}=\frac{12}{2}=6$$

$$2c=6 \Rightarrow c=3\in(2,4)$$

Hence mean value theorem verified.

41) Find the intervals in which the function f given by $f(x)=x^2-4x+6$ is

(i) strictly increasing

(ii) strictly decreasing.

Solution: $f(x)=x^2-4x+6,$

$f^1(x)=2x-4$

for strictlyincresing $f^1(x)>0,$

i.e., $2x-4>0 \Rightarrow x>2$

i) Strictly increasing $x\in(2,\infty)$

for strictly decresing $f^1(x)<0$

i.e., $2x-4<0 \Rightarrow x<2$

ii) Strictly decreasing $x\in(-\infty,2)$

42) Find $\int\frac{x}{(x+1)(x+2)}dx$

Solution: Let $\frac{x}{(x+1)(x+2)}=\frac{A}{(x+1)}+\frac{B}{(x+2)}$

$$\frac{x}{(x+1)(x+2)}=\frac{A(x+2)+B(x+1)}{(x+1)(x+2)}$$

$$x=A(x+2)+B(x+1)$$

Put $x=-2$ we get $B=2$

Put $x=-1$ we get $A=-1$

$$\int\frac{x}{(x+1)(x+2)}dx=\int\left(\frac{-1}{(x+1)}+\frac{2}{(x+2)}\right)dx$$

$$= 2\int \frac{1}{(x+2)}dx - \int \frac{1}{(x+1)}dx$$

$$= 2\log(x+2) - \log(x+1) + C$$

$$= \log(x+2)^2 - \log(x+1) + C$$

$$= \log\left|\frac{(x+2)^2}{(x+1)}\right| + C$$

43) Evaluate $\int_0^1 \frac{\tan^{-1}x}{1+x^2}.dx$

Solution: Let $I = \int_0^1 \frac{\tan^{-1}x}{1+x^2}\,dx$

put $\tan^{-1}x = t$

$$\frac{dx}{1+x^2} = dt$$

put $x = 0 \quad t = 0$

$x = 1 \qquad t = \pi/4$

$$\therefore I = \int_0^{\pi/4} t\,dt = \left.\frac{t^2}{2}\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left(\frac{\pi^2}{16}\right) = \frac{\pi^2}{32}$$

44) Find the integral of $\int \frac{(x-3)}{(x-1)^3}e^x$ **with respect to x**

Solution: $I = \int \frac{(x-3)}{(x-1)^3}e^x.dx$

$$= \int e^x\left(\frac{(x-1)-2}{(x-1)^3}\right)dx$$

$$= \int e^x\left(\frac{1}{(x-1)^2} - \frac{2}{(x-1)^3}\right)dx$$

Here $f(x) = \frac{1}{(x-1)^2} \Rightarrow f^1(x) = \frac{-2}{(x-1)^3}$

$$\int e^x\left(f(x) + f^1(x)\right)dx = e^x f(x) + c$$

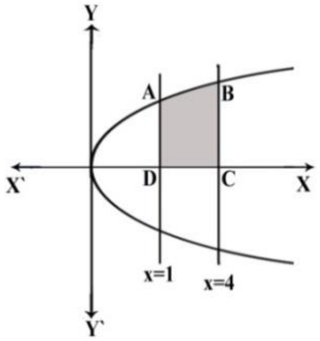
$$\therefore \qquad \int e^x\left(\frac{1}{(x-1)^2} - \frac{2}{(x-1)^3}\right)dx = \frac{e^x}{(x-1)^2} + c$$

45) **Determine the area of the region bounded by $y^2 = x$ and the lines x =1, x = 4 and the x-axis in the first quadrant.**

Solution : Given equation of the curve $y^2 = x$

$\therefore \quad y = \sqrt{x}$

$\therefore$ Required area of region ABCDA



$$A = \int_1^4 \sqrt{x}\,dx$$

$$= \left\{ \frac{x^{3/2}}{3/2} \right\}_1^4$$

$$= \frac{2}{3}\left[x\sqrt{x}\right]_1^4$$

$$= \frac{2}{3}\left[4\sqrt{4} - 1\sqrt{1}\right]$$

$$= \frac{2}{3}[4\times 2 - 1] = \frac{2}{3}[7]$$

$$= \frac{14}{3}\text{sq unit}$$

46) **Form the Differential Equation representing the family of parabolas having vertex at origin and axis along positive direction of x-axis.**

Solution: The eqn, of the family of parabola having the vertex at origin and the axis along the positive x-axis is $y^2 = 4ax$ -----(1)

Differentiating eqn (1) w. r. t. x, we get

$$2y\frac{dy}{dx} = 4a \text{ -----(ii)}$$

Substituting the value of 4a from eqn (ii) in

eqn(1) we get $\quad y^2 = \left(2y\frac{dy}{dx}\right)(x)$

$\Rightarrow \qquad y^2 - 2xy\frac{dy}{dx} = 0$

Which is the required D. E.

47) **Find the equation of the curve passing through the point (1, 1) whose differential equation is $\mathbf{x\,dy = (2x^2+1)dx\ [(x \neq 0)]}$**

Solution: The given differential equation can be expressed as

$$dy = \left(\frac{2x^2+1}{x}\right)dx \text{ ....}(1)$$

Integrate both sides of equation (1) we get

$$\int dy = \int \left(2x + \frac{1}{x}\right)dx$$

$$y = x^2 + \log|x| + C \text{..................}(2)$$

it passes through (1, 1) therefore substitute
(1, 1) in equation (2) we get c=0.
Now, substitute c = 0 in equation (2)
we get required equation of curve $y = x^2 + \log|x|$

48) Three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ satisfy the condition $\vec{a}+\vec{b}+\vec{c}=0$. Evaluate the quantity $\mu = \vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$, if $|\vec{a}|=1$ $|\vec{b}|=4$ and $|\vec{c}|=2$.

Solution: Given $\vec{a}+\vec{b}+\vec{c}=0$,

$$|\vec{a}|=1 \; |\vec{b}|=4 \text{ and } |\vec{c}|=2$$

$$\therefore \quad (\vec{a}+\vec{b}+\vec{c}).(\vec{a}+\vec{b}+\vec{c})=0$$

$$\Rightarrow \quad |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a})=0$$

$$\Rightarrow \quad 1+16+4+ \; 2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a})=0$$

$$\Rightarrow \quad 2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a})=-21$$

$$\Rightarrow \quad 2\mu=-21 \quad \Rightarrow \quad \mu=\frac{-21}{2}.$$

49) Prove that $[\vec{a},\vec{b},\vec{c}+\vec{d}]=[\vec{a},\vec{b},\vec{c}]+[\vec{a},\vec{b},\vec{d}]$

Solution: We have $[\vec{a},\vec{b},\vec{c}+\vec{d}]=\vec{a}.(\vec{b}\times(\vec{c}+\vec{d}))$

$$=\vec{a}.(\vec{b}\times\vec{c}+\vec{b}\times\vec{d})$$

$$=\vec{a}.(\vec{b}\times\vec{c})+\vec{a}.(\vec{b}\times\vec{d})$$

$$=[\vec{a},\vec{b},\vec{c}]+[\vec{a},\vec{b},\vec{d}]$$

50) Find equation of plane passing through line of intersection of the planes 3x - y + 2z – 4 = 0 and x + y + z -2=0 and the point (2, 2, 1).

Solution: Equation of plane passing through the line of intersection is,

$$(3x-y+2z-4)+\lambda(x+y+z-2)=0 \text{ ----}(1)$$

Given that, $(x,y,z)=(2,2,1)$

$$\therefore \quad (3(2)-2+2(1)-4)+\lambda[2+2+1-2]=0$$

$$2+\lambda(3)=0$$

$$\lambda(3)=-2 \quad \Rightarrow \quad \lambda=-2/3$$

$$\therefore (1)\Rightarrow (3x-y+2z-4)-\frac{2}{3}(x+y+z-2)=0$$

$$9x - 3y + 6z - 12 - 2x - 2y - 2z + 4 = 0$$
$$\Rightarrow 7x - 5y + 4z - 8 = 0$$

51) A fair coin and an unbiased die are tossed. Let A be the event 'head appears on the coin' and B be the event '3 on the die'. Check whether A and B are independent events or not.

Solution:

The sample space is given by, $S = \begin{Bmatrix} (H,1)(H,2)(H,3)(H,4)(H,5)(H,6) \\ (T,1)(T,2)(T,3)(T,4)(T,5)(T,6) \end{Bmatrix}$

Let A: Head appears on the coin

$A = \{(H,1)(H,2)(H,3)(H,4)(H,5)(H,6)\}$

$\Rightarrow P(A) = \frac{6}{12} = \frac{1}{2}$

B: 3 on die $= \{(H,3),(T,3)\}$

$\Rightarrow P(B) = \frac{2}{12} = \frac{1}{6}$

$A \cap B = \{(H,3)\}$

$P(A \cap B) = \frac{1}{12}$

$P(A) \times P(B) = \frac{1}{2} \times \frac{2}{6} = P(A \cap B)$

Therefore, A and B are independent events.

PART – D

Answer any six questions

52) Verify wheather the function $f: R \to R$ defined by $f(x) = 1 + x^2$ is one – one , onto or bijective. Justify your answer.

Solution: $f: R \to R$ defined by $f(x) = 1 + x^2$

$x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

$\Rightarrow 1 + x_1^2 = 1 + x_2^2$

$\Rightarrow x_1^2 = x_2^2$

$\Rightarrow x_1 = \pm x_2$

$\therefore$ $f(x_1) = f(x_2)$ does not imply that $x_1 = x_2$

Consider $f(1) = f(-1) = 2$

$\therefore$ f is not one-one

Consider an element – 2 in co domain R.

It is seen that $f(x) = 1 + x^2$ is positive for all $x \in R$.

$\therefore f$ is not onto.

Hence, f is neither one-one nor onto.

53) Show that the function $f : R \to R$ defined by f(x) = 4x+3 is invertible. Also find the inverse of f.

Solution : Consider an arbitrary elements y in R.

Given function is f(x) = 4x+3

for some x in the domain R. $\Rightarrow y = 4x + 3$

$$\Rightarrow \quad 4x = y - 3, \ \Rightarrow \ x = \frac{y-3}{4} \ \forall y \in R$$

Let us define the function $g : R \to R$

defrined by $g(y) = \frac{y-3}{4}$.

Now $fog(y) = f(g(y)) = f\left(\frac{y-3}{4}\right)$

$$= 4\left(\frac{y-3}{4}\right) + 3 = y$$

$\therefore \quad fog(y) = I_R$

And $gof(x) = g(4x+3) = \frac{4x+3-3}{4} = \frac{4x}{4} = x$

$\therefore \quad gof(x) = I_R$

This shows that $fog(y) = I_R$ and $gof(x) = I_R$

Hence the given function is invertible with $f^{-1} = g$.

Hence the given function is invertible.

Let $\quad f(x) = y \quad \Rightarrow \quad f^{-1}(y) = x.$

$$\Rightarrow \quad f^{-1}(y) = \frac{y-3}{4} \quad \because \quad f^{-1}(x) = \frac{x-3}{4}$$

54) If $A = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}$ **and** $C = \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$, **then compute** $(A+B)$ **and** $(B-C)$. **Also verify that A+(B - C) = (A+B)-C.**

Solution : $A + B = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 1 & -1 \\ 9 & 2 & 7 \\ 3 & -1 & 4 \end{bmatrix}$

$$B - C = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -2 & 0 \\ 4 & -1 & 3 \\ 1 & 2 & 0 \end{bmatrix}$$

$$A + (B - C) = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} -1 & -2 & 0 \\ 4 & -1 & 3 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -3 \\ 9 & -1 & 5 \\ 2 & 1 & 1 \end{bmatrix}$$

$$(A + B) - C = \begin{bmatrix} 4 & 1 & -1 \\ 9 & 2 & 7 \\ 3 & -1 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -3 \\ 9 & -1 & 5 \\ 2 & 1 & 1 \end{bmatrix}$$

$\therefore \quad$ A+ (B - C) = (A + B) -C.

55) Solve the following system of equations by matrix method
3x – 2y +3z = 8, 2x +y –z = 1 ,4x – 3y + 2z =4

Solution: Let $A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{vmatrix}$$

$$|A| = 3(2-3) + 2(4+4) + 3(-6-4)$$

$$|A| = -3 + 16 - 30$$

$$|A| = -17 \neq 0$$

Hence, A is non-singular and so its inverse exists.

Now,

$$\text{Co-factor of A} \quad \begin{matrix} A_{11} = -1 & A_{12} = -8 & A_{13} = -10 \\ A_{21} = -5 & A_{22} = -6 & A_{23} = 1 \\ A_{31} = -1 & A_{32} = 9 & A_{33} = 7 \end{matrix}$$

$$\text{Co-factor matrix A} = \begin{bmatrix} -1 & -8 & -10 \\ -5 & -6 & 1 \\ -1 & 9 & 7 \end{bmatrix}$$

$$\therefore \quad \text{adj}A = \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|}\text{adj}(A)$$

$$A^{-1} = \frac{1}{-17}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$$

Given system of equations can be written as

$$AX = B \quad X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-17}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}\begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{-1}{17}\begin{bmatrix} -17 \\ -34 \\ -51 \end{bmatrix}$$

$$\therefore \quad x = 1; \quad y = 2; \quad z = 3$$

56) If $y = (\tan^{-1}x)^2$ then show that $(x^2+1)^2\frac{d^2y}{dx^2} + 2x(x^2+1)\frac{dy}{dx} = 2.$

Solution: $y = (\tan^{-1}x)^2$

Differentiate.w.r.t.x

$$\frac{dy}{dx} = 2(\tan^{-1}x) \times \frac{1}{1+x^2}$$

cross multplying

$$\left(1+x^2\right)\frac{dy}{dx}=2\left(\tan^{-1}x\right)$$

Again Diff.w.r.t. x on both sides

$$\left(1+x^2\right)\frac{d^2y}{dx^2}+\frac{dy}{dx}.2x=\frac{2}{1+x^2}$$

multiply $\left(1+x^2\right)$ on bothsides

$$\left(1+x^2\right)^2\frac{d^2y}{dx^2}+2x\left(1+x^2\right)\frac{dy}{dx}=2$$

57) If length of x rectangle is decreasing at the rate of 3cm/ minute and the width y is increasing at the rate of 2cm/ minute, when x =10cm and y = 6cm. Find the rate of change of (i) the perimeter (ii) the area of the rectangle.

Solution: Given $\frac{dx}{dt}=-3cm/min \quad \frac{dy}{dt}=+2cm/min$

When x = 10cm, y = 6cm.

(i) $p=2(x+y)$

D.w.r. to t

$$\frac{dp}{dt}=2\left(\frac{dx}{dt}+\frac{dy}{dt}\right)$$

$$\Rightarrow \quad \frac{dp}{dt}=2(-3+2)=-2cm/min$$

$\therefore$ Perimeter is decreasing at the rate of2cm/min.

(ii) $A=x.y$

$$\Rightarrow \quad \frac{dA}{dt}=x.\frac{dy}{dt}+y.\frac{dx}{dt}$$

$$=10(2)+6(-3)=20-18$$

$$=2cm^2/min$$

$\therefore$ Area is increasing at the rate of $2cm^2$/min.

58) Find $\int\frac{dx}{x^2-a^2}$ Hence evaluate $\int\frac{dx}{x^2-16}$

Solution: we have $\frac{1}{x^2-a^2}=\frac{1}{(x-a)(x+a)}$

$$=\frac{1}{2a}\left[\frac{(x+a)-(x-a)}{(x+a)(x-a)}\right]$$

$$=\frac{1}{2a}\left[\frac{1}{x-a}-\frac{1}{x+a}\right]$$

$$\therefore \quad \int\frac{dx}{x^2-a^2}=\frac{1}{2a}\left[\int\frac{dx}{(x-a)}-\int\frac{dx}{(x+a)}\right]$$

$$=\frac{1}{2a}\left[\log|x-a|-\log|x+a|\right]+c$$

$$= \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$$

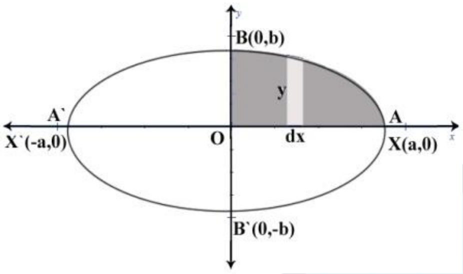
By the above result,

$$\int \frac{dx}{x^2 - 16} = \int \frac{dx}{x^2 - 4^2}$$

$$\frac{1}{2\times 4}\log\left|\frac{x-4}{x+4}\right| + c = \frac{1}{8}\log\left|\frac{x-4}{x+4}\right| + c$$

59) Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ by using integration

Solution:



Area of ellipse = 4 (Area of the region ABOA).

$$\text{Area of region ABOA} = \int_0^a y\,dx$$

$$\left[\text{Now, } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right.$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\frac{y^2}{b^2} = \frac{a^2 - x^2}{a^2}$$

$$y^2 = \frac{b^2}{a^2}\left(a^2 - x^2\right)$$

$$\left. y = \frac{b}{a}\sqrt{a^2 - x^2} \quad \right]$$

$$\text{Area of region ABOA} = \frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$$

$$= \frac{b}{a}\left(\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right)_0^a$$

$$= \frac{b}{a}\left[\left(\frac{a}{2}\times 0 + \frac{a^2}{2}\sin^{-1}1\right) - 0\right]$$

$$= \frac{b}{a}\left(\frac{\pi a^2}{4}\right)$$

$$\therefore \quad \text{Area of ellipse} = 4\left(\frac{b}{a}\right)\frac{\pi a^2}{4} = \pi ab$$

60) Find the general solution of the differential equation $\cos^2 x.\frac{dy}{dx} + y = \tan x\left(0 \le x < \frac{\pi}{2}\right)$

Solution: We have $\cos^2 x.\frac{dy}{dx} + y = \tan x\left(0 \le x < \frac{\pi}{2}\right)$

divided by $\cos^2 x$ we get

$$\frac{dy}{dx} + y.\sec^2 x = \tan x.\sec^2 x$$

compare with $\frac{dy}{dx} + py = Q$

$p = \sec^2 x \qquad Q = \tan x.\sec^2 x$

$$I.F = e^{\int p.dx} = e^{\int \sec^2 x.dx} = e^{\tan x}$$

solution of differential equation is

$$y(I.F) = \int Q(I.F).dx + c$$

$$y.e^{\tan x} = \int \tan x.\sec^2 x.e^{\tan x}.dx + c$$

$$y.e^{\tan x} = I + C\text{---------}(1)$$

when $I = \int e^{\tan x}.\tan x.\sec^2 x.dx$

put $\tan x = t \quad \Rightarrow \quad \sec^2 x.dx = dt$

$$I = \int e^t.t.dt$$

$$I = t.e^t - \int e^t.dt$$

$$I = t.e^t - e^t$$

$$I = \tan x.e^{\tan x} - e^{\tan x}\text{--------}(2)$$

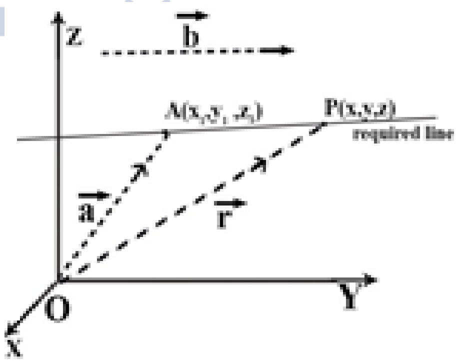
substitute (2) in (1)

$$y.e^{\tan x} = \tan x.e^{\tan x} - e^{\tan x} + c$$

$$y = \tan x - 1 + c.e^{-\tan x}$$

61) Derive the equation of line in space passing through a point and parallel to the vector both in vector and Cartesian form.

Solution: Let $\vec{a}$ be the position vector of the given point A with respect to the origin O. let 'l' be the line passes through the point A and is parallel to a given vector $\vec{b}$. Let $\vec{r}$ be the position vectors of any point P on the line.



Then $\overrightarrow{AP}$ is parallel to $\vec{b}$,

We have $\overrightarrow{AP} = \lambda\vec{b}$

$\Rightarrow \quad \overrightarrow{OP} - \overrightarrow{OA} = \lambda\vec{b}$

$\Rightarrow \quad \vec{r} - \vec{a} = \lambda\vec{b}$

$\Rightarrow \quad \vec{r} = \vec{a} + \lambda\vec{b}$

This gives the position vector of any point P on the line.

Hence it is called vector equation of the line.

Cartesian form:

Let the coordinates of the given points

$A(x_1, y_1, z_1)$ and the direction ratios of the line be a , b, c. consider the coordinates of any point P(x, y, z).

Then $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and

$\vec{b} = a\hat{i} + b\hat{j} + c\hat{k}$

Substituting in $\vec{r} = \vec{a} + \lambda\vec{b}$, we get

$x\hat{i} + y\hat{j} + z\hat{k} = \left(x_1\hat{i} + y_1\hat{j} + z_1\hat{k}\right) + \lambda\left(a\hat{i} + b\hat{j} + c\hat{k}\right)$

$x\hat{i} + y\hat{j} + z\hat{k} = \left(x_1 + a\lambda\right)\hat{i} + \left(y_1 + b\lambda\right)\hat{j} + \left(z_1 + c\lambda\right)\hat{k}$ equating their components.

We get $x = x_1 + a\lambda,\ y = y_1 + b\lambda,\ z = z_1 + c\lambda \Rightarrow \quad \lambda = \frac{x - x_1}{a},\ \ \lambda = \frac{y - y_1}{b},\ \ \lambda = \frac{z - z_1}{c}$

$\Rightarrow \frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$

This is the Cartesian equation of the line.

62) A bag contains 4 red and 4 black balls another bag contains 2 red and 6 black balls. One of the bag is selected at random and ball is drawn from the bag which is found to be red. Find the probability that the ball is drawn from first bag.

Solution: E_1 : Event of choosing bag I; $P\left(E_1\right) = \frac{1}{2}$

E_2 : Event of choosing bag II; $P\left(E_2\right) = \frac{1}{2}$

A : Ball drawn is a red ball

$P\left(A \mid E_1\right) = \frac{4}{8} = \frac{1}{2}$

$P\left(A \mid E_2\right) = \frac{2}{8} = \frac{1}{4}$

By Baye's theorem

$$P\left(E_1 \mid A\right) = \frac{P\left(E_1\right)P\left(A \mid E_1\right)}{P\left(E_1\right)P\left(A \mid E_1\right) + P\left(E_2\right)P\left(A \mid E_2\right)}$$

$$= \frac{\frac{1}{2}\times\frac{1}{2}}{\frac{1}{2}\times\frac{1}{2} + \frac{1}{2}\times\frac{1}{4}} = \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{8}} = \frac{\frac{1}{4}}{\frac{3}{8}}$$

$P\left(E_1 \mid A\right) = \frac{2}{3}$

63) If a fair coin is tossed 10 times. Find the probability of

a) exactly six heads

b) at least six head.

Solution: Let X be the number of heads obtained when a fair coin is tossed 10 times.

Now, n=10

P = p (getting a head) $= \frac{1}{2}$

q = 1-p $= 1 - \frac{1}{2} = \frac{1}{2}$

The binomial distribution of x is

$P(X=x) = {}^{n}c_{x} p^{x} q^{n-x}$ where x = 0,1,2…..n

$\therefore \quad P(X=x) = {}^{10}c_{x}\left(\frac{1}{2}\right)^{x}\left(\frac{1}{2}\right)^{10-x}$

a) P(Exactly six heads) =

$$P(X=6) = {}^{10}c_{6}\left(\frac{1}{2}\right)^{6}\left(\frac{1}{2}\right)^{10-6}$$

$$= 210\times\frac{1}{64}\times\frac{1}{16} = \frac{210}{1024} = \frac{105}{512}$$

b) P (At least 6 heads) = $P(x\geq 6)$

$= P(x=6)+P(x=7)+P(x=8)+P(x=9)$

$+P(x=10)$

$$= {}^{10}c_{6}\left(\frac{1}{2}\right)^{10} + {}^{10}c_{7}\left(\frac{1}{2}\right)^{10} + {}^{10}c_{8}\left(\frac{1}{2}\right)^{10} + {}^{10}c_{9}\left(\frac{1}{2}\right)^{10}$$

$$+{}^{10}c_{10}\left(\frac{1}{2}\right)^{10}$$

$$= \frac{105}{512}+\frac{120}{1024}+\frac{45}{1024}+\frac{10}{1024}+\frac{1}{1024}$$

$$= \frac{386}{1024} = \frac{193}{512}$$

PART – E

Answer ANY ONE question

64)

a) Maximize Z = 4x + y

Subject to the constraints: $\mathbf{x+y\leq 50;\ 3x+y\leq 90;\quad x\geq 0; y\geq 0}$

Solution: We have to minimize Z = 4x +y

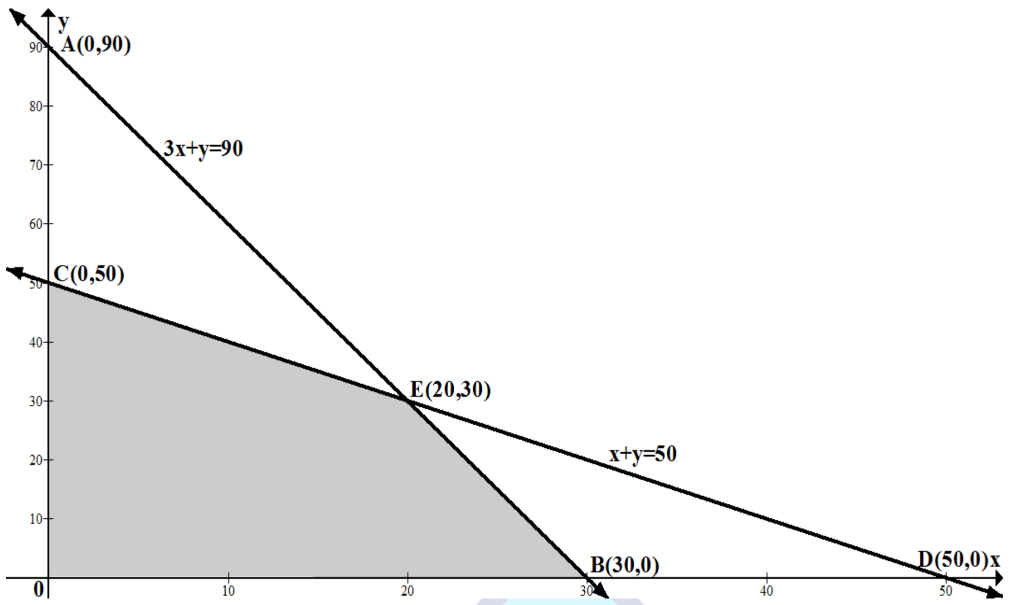
Now changing the given in equation $x+y\leq 50$ ––––––––––(1)

$3x+y\leq 90$ ––––––(2) $x,y\geq 0$ ––––––––(3)

To equation,

$3x+y=90$		
x	0	30
y	90	0

$x+y=50$		
x	0	50
y	50	0



The shaded region in the above fig is feasible region determined by the system of constraints (1) to (3). It is observed that the feasible region is bounded. The coordinates of the corner point OBEC are (0, 0), (30, 0) (20 , 30) and (0, 50)

The maximum value of Z = 4x +y

Corner point	Z = 4x + y
(0 , 0)	Z = 0
(30 , 0)	Z = 120 **maximum**
(20 ,30)	Z = 110
(0 , 50)	Z = 50

$\because \quad Z_{maxi} = 120$ at the point $(30, 0)$

b) **If the matrix** $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ **satisfies the equation** $A^2 - 4A + I = O$, **where I is** 2×2 **identify matrix and O is** 2×2 **zero matrix. Using this equation, find A**$^{-1}$**.**

Solution:

Now, $A^2 - 4A + I = O$

Therefore, $AA - 4A = -I$

or $AA(A^{-1}) - 4AA^{-1} = -IA^{-1}$ (Post multiplying by A^{-1} because $|A| \neq 0$)

or $A(AA^{-1}) - 4I = -A^{-1}$

or $AI - 4I = -A^{-1}$

or $A^{-1} = 4I - A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Hence $A^{-1} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

65) a) Prove that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ and hence evaluate $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$

Consider $\int_0^a f(a-x)dx$

Put $a-x=t$ in RHS
$dx=-dt$

When $x=a,\ t=0,$
and $x=0,\ t=a$

a) $= -\int_a^0 f(t)(dt)$

$= \int_0^a f(t)dt \quad \left[\because \int_0^a f(x\,dx) = -\int_a^0 f(x)dx\right]$

$\int_0^a f(t)dx = \int_0^a f(x)dx$

$\therefore \int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$\mathbf{I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx \ ....(1)}$$

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin\left(\pi/2 - x\right)}}{\sqrt{\sin\left(\pi/2 - x\right)}+\sqrt{\cos\left(\pi/2 - x\right)}}dx$$

$$I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx ....(2)$$

$(1)+(2)$ gives

$$2I = \int_0^{\pi/2} \left(\frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\right)dx$$

$$= \int_0^{\pi/2} 1\,dx$$

$$2I = [x]_0^{\pi/2}$$

$$2I = \frac{\pi}{2}$$

$$\therefore \quad I = \frac{\pi}{4}.$$

b) Find the value of K, if $f(x) = \begin{cases} Kx+1, & \text{if } x \le \pi \\ \cos x, & \text{if } x > \pi \end{cases}$ **is continuous at** $x=\pi$.

Solution: The function is defined at $x=\pi$.

i.e., $f(x) = Kx+1 \Rightarrow f(\pi) = K\pi+1$

$LHL = \lim_{x\to\pi^-} f(x) = \lim_{x\to\pi+} Kx+1 = K\pi+1$

$RHL = \lim_{x\to\pi^-} Kx+1 = \lim_{x\to\pi+} \cos x = \cos\pi$

f is continuous at $x=\pi$

$\therefore \quad K\pi+1=\cos\pi$

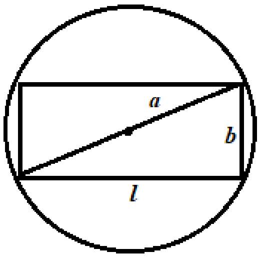
$$K\pi + 1 = -1$$

$$K = \frac{-2}{\pi}$$

66.

a) Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.

Solution: Let a rectangle of length l and breadth b be inscribed in the given circle of radius a. Then, the diagonal passes through the centre and is of length $2a$ cm.



Now, by applying the Pythagoras theorem, we have:

$$(2a)^2 = l^2 + b^2$$

$$\Rightarrow b^2 = 4a^2 - l^2$$

$$\Rightarrow b = \sqrt{4a^2 - l^2}$$

Area of triangle, $A = l\sqrt{4a^2 - l^2}$

Therefore, $\frac{dA}{dl} = \sqrt{4a^2 - l^2} + l\frac{1}{2\sqrt{4a^2 - l^2}}(-2l)$

$$= \sqrt{4a^2 - l^2} - \frac{l^2}{\sqrt{4a^2 - l^2}}$$

$$= \frac{4a^2 - 2l^2}{\sqrt{4a^2 - l^2}}$$

$$\frac{d^2A}{dl^2} = \frac{\sqrt{4a^2 - l^2}(-4l) - (4a^2 - 2l^2)\frac{(-2l)}{2\sqrt{4a^2 - l^2}}}{4a^2 - l^2}$$

$$= \frac{(4a^2 - l^2)(-4l) + l(4a^2 - l^2)}{(4a^2 - l^2)^{\frac{3}{2}}}$$

$$= \frac{-12a^2 l + 2l^3}{(4a^2 - l^2)^{\frac{3}{2}}}$$

$$= \frac{-2l(6a^2 - l^2)}{(4a^2 - l^2)^{\frac{3}{2}}}$$

Now, $\frac{dA}{dl} = 0$

Hence, $\Rightarrow \frac{4a^2 - 2l^2}{\sqrt{4a^2 - l^2}} = 0$

$$\Rightarrow 4a^2 = 2l^2$$

$$\Rightarrow l = \sqrt{2}a$$

Thus, $b=\sqrt{4a^2-2a^2}$

$=\sqrt{2a^2}$

$=\sqrt{2}a$

When, $l=\sqrt{2}a$

Then, $\dfrac{d^2A}{dl^2}=\dfrac{-2\left(\sqrt{2}a\right)\left(6a^2-2a^2\right)}{2\sqrt{2}a^3}$

$=\dfrac{-8\sqrt{2}a^3}{2\sqrt{2}a^3}$

$=-4<0$

By the second derivative test, when $l=\sqrt{2}a$, then the area of the rectangle is the maximum.

Since, $l=b=\sqrt{2}a$ the rectangle is a square.

Hence, it has been proved that of all the rectangles inscribed in the given fixed circle, the square has the maximum area.

b) Prove that $\begin{vmatrix} \mathbf{1} & \mathbf{a} & \mathbf{a^2} \\ \mathbf{1} & \mathbf{b} & \mathbf{b^2} \\ \mathbf{1} & \mathbf{c} & \mathbf{c^2} \end{vmatrix} = \mathbf{(a-b)(b-c)(c-a)}$**.**

Solution: L.H.S. $=\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

By applying $R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$

$$=\begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$=\begin{vmatrix} 0 & a-b & (a+b)(a-b) \\ 0 & b-c & (b+c)(b-c) \\ 1 & c & c^2 \end{vmatrix}$$

Taking (a – b) and (b – c) common from R_1 and R_2 respectively.

$$=(a-b)(b-c)\begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

Expanding along R_1

$= (a - b)(b - c)[0(c^2 - bc - c^2) - 1(0 - b - c) + (a + b)(0 - 1)]$

$= (a - b)(b - c)[0 + b + c - a - b]$

$= (a - b)(b - c)(c - a)$

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DEPARTMENT OF MATHEMATICS

- Mr. Ashwath . S. L
- Mr. Ganapathi K. S
- Mr. Nandeesh H. B
- Mr. Adithya Vati K
- Mr. Rakshith B. S
- Mr. Sumanth Damle

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