2023-24 II PUC ANNUAL EXAMINATION

MATHEMATICS

PART – A

I. Answer ALL the multiple choice questions **(15 × 1 = 15)**

1. The relation R in the se {1, 2, 3} given by R = {(1, 1), (2, 2), (1, 2), (2, 3), (3, 3)} is:

A) Reflexive **B) Reflexive and Symmetric**

B) Reflexive and Transitive **D) Symmetric and Transitive**

Ans: A)

R is reflexive only

Since (1,2) belongs to R but (2,1) not in R so R is not symmetric

Also (1,2) ,(2,3) in R but (1,3) not in R so R is not transitive

2. If f: Z → Z, where Z is the set of integers is defined as f(x) = 3x then

A) f is both one-one and onto **B) f is many one and onto**

C) f is one-one but onto **D) f is neither one-one nor onto**

Ans: C)

f is one to one but f is not onto

As $x=\frac{y}{3}$ not belongs to Z

3. The principal value branch of $\sin^{-1} x$ is

A) $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ **B) (0, π)** **C) [0, π]** **D) [0, 2π]**

Ans: A)

Principal branch value of $\sin^{-1}x$ is $[-\frac{\pi}{2},\frac{\pi}{2}]$

4. If A = [a_{ij}] is a 2 × 2 matrix whose elements are given by $a_{ij} = \frac{i}{j}$ then A is

A) $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ **B)** $\begin{bmatrix} 1 & \frac{1}{2} \\ 2 & 1 \end{bmatrix}$ **C)** $\begin{bmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{bmatrix}$ **D)** $\begin{bmatrix} 1 & 2 \\ \frac{1}{2} & 1 \end{bmatrix}$

Ans: B)

$a_{11}=1$, $a_{12}=1/2$, $a_{21}=2$, $a_{22}=1$

5. If A is an invertible matrix of order 2 × 2 det (A) = 5 then det (A^{-1}) is equal to

A) 5 **B)** $\frac{1}{25}$ **C)** $\frac{1}{5}$ **D) 25**

Ans: C)

Det(A^{-1})=$\frac{1}{|A|} = \frac{1}{5}$

6. The function f: R → R defined as f(x) = [x], where [x] denotes the greatest integer less than or equal to x. For what values of x in the interval 2 < x < 5 given below f(x) is not differentiable?

A) 2 and 5 **B) 3 and 5** **C) 4 and 5** **D) 3 and 4**

Ans: D)

greatest integer function is not continuous in integral values of x . So it is not differentiable at x=3 and x=4

7. If $y=\sin\left(x^2+5\right)$**, then** $\frac{dy}{dx}$ **is**

A) $\cos(x^2+5)$ **B)** $-2x\cos(x^2+5)$ **C)** $\cos\left(x^2+5\right)\left(2x+5\right)$ **D)** $2x\cos(x^2+5)$

Ans: D)

y=sin (x^2+5) , derivative is 2x cos(x^2+5)

8. The maximum value of the function $f(x)=x, \quad x\in(1,2)$ **is**

A) 1 **B) do not have maximum value**

C) 3 **D) 2**

Ans: B)

Since x∈ (1,2) f has neither maxima nor minima

9. $\int \sec x(\sec x+\tan x)dx$ **is**

A) $\sec^2 x+\tan x+c$ **B)** $\sec x+\tan x+c$ **C)** $\sec x-\tan x+c$ **D)** $-\tan x-\sec x+c$

Ans: B)

$$\int \sec x(\sec x+\tan x)dx=\int \sec^2 x+\sec x\tan x\,dx=\tan x+\sec x+c$$

10. $\int e^x(\sin x+\cos x)dx$ **is equal to**

A) $e^x\cos x+c$ **B)** $e^x\tan x+c$ **C)** $e^x\sin x+c$ **D)** $-e^x\cos x+c$

Ans: C)

here f(x)=sinx and f^1(x)=cosx by using the property

$\int e^x\left(f(x)+f^1(x)\right)=e^x f(x)+C$, $\int e^x(\sin x+\cos x)=e^x\sin x+c$

11. The projection of the vector $\vec{a}=\hat{i}+3j+7k$ **along x-axis is**

A) 1 **B) 3** **C) 7** **D) 0**

Ans: A)

The projection of vector $\vec{a}=\hat{\imath}+3\hat{\jmath}+7\hat{k}$ is equal to the scalar component 1

12. The unit vector in the direction of $\vec{a}=\hat{i}+j+2k$ **is**

A) $\frac{\hat{i}-j-2k}{6}$ **B)** $\frac{\hat{i}+j+2k}{\sqrt{6}}$ **C)** $\frac{\hat{i}-j+2k}{6}$ **D)** $\frac{\hat{i}+j-2k}{\sqrt{6}}$

Ans: B)

Unit vector, $\hat{a}=\frac{1}{|\vec{a}|}\vec{a}=\frac{1}{\sqrt{6}}\left(\hat{\imath}+\hat{\jmath}+2\hat{k}\right)$

13. If a line makes $90^o, 135^o, 45^o$ **with the x, y and z axes respectively then direction cosines are**

A) $0,\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}}$ **B)** $0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}$ **C)** $1,\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}}$ **D)** $1,\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}}$

Ans: A)

Direction cosines are $\cos 90°, \cos(135°), \cos 45° = 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

14. Which of the following is a non-negative constraints in a Linear Programming Problem?

A) $x \geq 0, y \leq 0$ **B)** $x \leq 0, y \leq 0$ **C)** $x \geq 0, y \geq 0$ **D)** $x \leq 0, y \geq 0$

Ans: C)

Non-negative constraints are $x \geq 0, y \geq 0$

15. If two cards are drawn without replacement from a pack of 52 playing cards then the probability that both the cards are black is

A) $\frac{1}{26}$ **B)** $\frac{1}{4}$ **C)** $\frac{25}{104}$ **D)** $\frac{25}{102}$

Ans: D)

Let P(AB) be the probability that both cards are black

$$P(AB) = P(A)P(B|A) = \frac{26}{52} \cdot \frac{25}{51} = \frac{25}{102}$$

II. Fill in the blanks by choosing the appropriate answer from those given in the bracket: (5×1 = 5)

$$\left(0,\ 1,\ 2,\ 3,\ \frac{1}{2},\ 6\right)$$

16. The value of $\sin\left(\operatorname{cosec}^{-1} 2\right)$ **is ____**

Ans: $\sin(cosec^{-1}2) = \sin\left(sin^{-1}\frac{1}{2}\right) = \frac{1}{2}$

17. If A is a square matrix of order 2×2 *and* $|A| = 8$ **then** $\left|\frac{1}{2}A\right|$ **is ____**

Ans: $\left|\frac{1}{2}A\right| = \frac{1}{4}|A| = \frac{1}{4} \cdot 8 = 2$

18. The order of the differential equation $\frac{d^3y}{dx^3} + y^2 + e^{\frac{dy}{dx}} = 0$ **is ___**

Ans: 3

19. Two lines with direction ratios 1,3,5 and 2, K, 10 are parallel then the value of K is ___

Ans: since lines are parallel, we have

$$\frac{1}{2} = \frac{3}{K} = \frac{5}{10} \Longrightarrow K = 6$$

20. If F is an event of the sample space S and $P(F) \neq 0$ **then** $P(S/F)$ **is ___**

Ans: $P(S|F) = \frac{P(S\cap F)}{P(F)} = \frac{P(F)}{p(F)} = 1$

PART-B

Answer any six questions: **(6 × 2 = 12)**

21. Show that $\sin^{-1}\left(2x\sqrt{1-x^2}\right)=2\sin^{-1}x,\ -\frac{1}{\sqrt{2}}\le x\le\frac{1}{\sqrt{2}}$.

Solution:

$$\text{LHS} = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$$

$$\text{Let } \sin\theta = x \quad \Rightarrow \quad \theta = \sin^{-1}x$$

$$= \sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$$
$$= \sin^{-1}\left(2\sin\theta\sqrt{\cos^2\theta}\right)$$

$$= \sin^{-1}\left(2\sin\theta\ \cos\theta\right)$$
$$= \sin^{-1}\left(\sin 2\theta\right) = 2\theta = 2\sin^{-1}x = \text{RHS}$$

22. Find the equation of the line joining the points (3,1) and (9,3) using determinants

Solution:

$$\text{Let}\left(x_1,y_1\right)=(3,1),\left(x_2,y_2\right)=(9,3),\left(x_3,y_3\right)=(x,y)$$

$$\text{Area of }\Delta=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}$$

$$\text{Area of }\Delta=\frac{1}{2}\begin{vmatrix} 3 & 1 & 1\\ 9 & 3 & 1\\ x & y & 1\end{vmatrix}$$

$$0=\frac{1}{2}\left[3(3-y)-1(9-x)+1(9y-3x)\right]$$
$$0=\frac{1}{2}\left[9-3y-9+x+9y-3x\right]$$
$$0=\frac{1}{2}\left[6y-2x\right]$$

$$0=3y-x$$
$$x-3y=0 \qquad \text{or} \qquad x=3y$$

23. If $2x+3y=\sin y$ **then find** $\frac{dy}{dx}$.

Solution:

$$\frac{d(2x)}{dx}+\frac{d(3y)}{dx}=\frac{d(\sin y)}{dx}$$
$$2+3\frac{dy}{dx}=\cos y\times\frac{dy}{dx}$$
$$\cos y\times\frac{dy}{dx}-3\frac{dy}{dx}=2$$

$$\frac{dy}{dx}(\cos y-3)=2$$

$$\frac{dy}{dx}=\frac{2}{(\cos y-3)}$$

24. The radius of a circle is increasing at the rate of 0.7 cm/s. what is the rate of increase of its circumference?

Solution:

Given that

$$\frac{dr}{dt}=0.7cm/s$$

To find $\frac{dC}{dt}$

We have circumference of the circle is $C=2\pi r$

Differentiate.w.r.to t,

$$\frac{dC}{dt}=2\pi.\frac{dr}{dt}=2\pi(0.7)=1.4\pi\ \ cm/s$$

25. Find the interval in which the function *f* given by $f(x)=2x^2-3x$ is decreasing.

Solution:

we have $f(x)=2x^2-3x$

$f^1(x)=4x-3$ for strictly decreasing $f^1(x)<0 \Rightarrow 4x-3<0$

i.e., $x<\frac{3}{4} \Rightarrow x\in\left(-\infty, \frac{3}{4}\right)$

26. Find $\int\frac{x}{(x+1)(x+2)}dx$.

Solution:

Let $\frac{x}{(x+1)(x+2)}=\frac{A}{(x+1)}+\frac{B}{(x+2)}$

$\Rightarrow x=A(x+2)+B(x+1)$

Equating the coefficients of x and constant term, we get

$A+B=1$

$2A+B=0$

On solving, we get

$A=-1$ and $B=2$

$$\therefore\frac{x}{(x+1)(x+2)}=\frac{-1}{(x+1)}+\frac{2}{(x+2)}$$

$$\Rightarrow\int\frac{x}{(x+1)(x+2)}dx=\int\frac{-1}{(x+1)}+\frac{2}{(x+2)}dx$$

$$=-\log|x+1|+2\log|x+2|+C$$

$$=\log(x+2)^2-\log(x+1)+C$$

$$=\log\frac{(x+2)^2}{(x+1)}+C$$

27. Evaluate $\int_1^{\sqrt{3}} \frac{dx}{1+x^2}$.

Solution:

Let

$$I = \int_1^{\sqrt{3}} \frac{dx}{1+x^2}$$

$$= \tan^{-1}x\Big]_1^{\sqrt{3}}$$

$$= \tan^{-1}\sqrt{3} - \tan^{-1}(1)$$

$$= \pi/3 - \pi/4 = \frac{4\pi - 3\pi}{12} = \frac{\pi}{12}$$

28. Consider two points P and Q with position vectors $\overrightarrow{OP} = 3\vec{a} - 2\vec{b}$ and $\overrightarrow{OQ} = \vec{a} + \vec{b}$. Find the position vector of a point R which divides the line joining P and Q internally in the ratio 2:1.

Solution:

The position vector of the point R

dividing the join of P and Q internally in the ratio 2 : 1 is

$$\overrightarrow{OR} = \frac{2(\vec{a} + \vec{b}) + (3\vec{a} - 2\vec{b})}{2+1} = \frac{5\vec{a}}{3}$$

29. Find the angle between the lines $\frac{x}{2} = \frac{y}{2} = \frac{z}{1}$ **and** $\frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8}$.

Solution:

The direction ration of the first line are $2, 2, 1$ and the direction ratios of the second line are $4, 1, 8$.

If θ is the angle between them, then

$$\cos\theta = \left| \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} \right|$$

$$\cos\theta = \left| \frac{2(4) + 2(1) + (1)(8)}{\sqrt{2^2 + 2^2 + 1^2}\sqrt{4^2 + 1^2 + 8^2}} \right|$$

$$\cos\theta = \left| \frac{18}{\sqrt{4+4+1}\sqrt{16+1+64}} \right|$$

$$= \left| \frac{18}{\sqrt{9}\sqrt{81}} \right| = \left| \frac{18}{27} \right| = \left| \frac{2}{3} \right|$$

$$\theta = \cos^{-1}\left(\frac{2}{3}\right)$$

30. Two coins are tossed once, where the events E and F are defined as

E: Tail appears on one coin

F : One coin shows Head

Find P(E/F)

Solution:

$S=\{HH,HT,TH,TT\}$

$(i) E=F=\{HT,TH\}$

$\therefore E\cap F=\{HT,TH\}$

$P(F)=\frac{2}{4}=\frac{1}{2}$

$P(E\cap F)=\frac{2}{4}=\frac{1}{2}$

$\therefore P(E/F)=\frac{P(E\cap F)}{P(F)}=1$

31. Let A and B be 2 independent events such that P(A) = 0.3 and P(B) = 0.6. Find.

a) P(A and not B)

b) P(neither A nor B)

Solutions:

Given, P(A) = 0.3 , P(B) = 0.6

a) P(A and not B) $=P(A\cap B')=P(A).P(B')$

$$=P(A)\cdot[1-P(B)]$$
$$=(0.3)[1-0.6]$$
$$=0.3\times 0.4=0.12$$

b) P(neither A nor B) $=P(A'\cap B')=P(A')\cdot P(B')$

$$=[1-P(A)][1-P(B)]$$
$$=[1-0.3][1-0.6]$$
$$=(0.7)\times(0.4)=0.28$$

PART – C

Answer any six questions **(6 × 3 = 18)**

32. Show that the relation R in R defined as $R=\{(a,b):a\le b\}$, is reflexive and transitive but not symmetric.

Solution:

(i) Reflexive

Let $a\in R$, $a\le a$ which is true

$\therefore$ $(a,a)\in R$ Thus, R is reflexive

(ii) Symmetric

Let $a,b\in R$, and $(a,b)\in R$,

Consider $a\le b$ does not imply $b\le a$

$\Rightarrow$ $(a,b)\in R$, but $(b,a)\notin R$

$\therefore$ R is not symmetric.

(iii) Transitive

Let $a,b,c\in R$, If $(a,b)\in R$, $\Rightarrow a\le b$

and (b,c) $\in$ R, $\Rightarrow b \le c \quad \Rightarrow$ (a,c) $\in$ R, $\Rightarrow a \le c$

Thus, R is transitive

Hence, R is reflexive, transitive but not symmetric.

33. Write $\tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right)$ $|x|<a$. in the simplest form.

Solution:

$$\tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right)=\tan^{-1}\left(\frac{a\sin\theta}{\sqrt{a^2-a^2\sin^2\theta}}\right)$$

Put $x=a\sin\theta \quad \Rightarrow \quad \theta=\sin^{-1}\frac{x}{a}$

$$=\tan^{-1}\left(\frac{a\sin\theta}{\sqrt{a^2\left(1-\sin^2\theta\right)}}\right)$$

$$=\tan^{-1}\left(\frac{a\sin\theta}{a\cos\theta}\right)$$

$$=\tan^{-1}\left(\tan\theta\right)$$

$$=\theta=\sin^{-1}x/a$$

34. Express $A=\begin{bmatrix}1 & 5\\ 6 & 7\end{bmatrix}$, as the sum of symmetric and skew symmetric matrix.

Solution:

Let $A=\begin{bmatrix}1 & 5\\ 6 & 7\end{bmatrix}$, $\quad A'=\begin{bmatrix}1 & 6\\ 5 & 7\end{bmatrix}$

$$A+A'=\begin{bmatrix}1 & 5\\ 6 & 7\end{bmatrix}+\begin{bmatrix}1 & 6\\ 5 & 7\end{bmatrix}=\begin{bmatrix}2 & 11\\ 11 & 14\end{bmatrix}$$

Let $P=\frac{1}{2}\left(A+A'\right)=\frac{1}{2}\begin{bmatrix}2 & 11\\ 11 & 14\end{bmatrix}=\begin{bmatrix}1 & \frac{11}{2}\\ \frac{11}{2} & 7\end{bmatrix}$

Now $P'=\begin{bmatrix}1 & \frac{11}{2}\\ \frac{11}{2} & 7\end{bmatrix}=P$

Thus $\quad P=\frac{1}{2}\left(A+A'\right)$ is a symmetric matrix

$$\left(A-A'\right)=\begin{bmatrix}1 & 5\\ 6 & 7\end{bmatrix}-\begin{bmatrix}1 & 6\\ 5 & 7\end{bmatrix}=\begin{bmatrix}0 & -1\\ +1 & 0\end{bmatrix}$$

Let $\quad Q=\frac{1}{2}\left(A-A'\right)=\frac{1}{2}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}0 & -\frac{1}{2}\\ \frac{1}{2} & 0\end{bmatrix}$

$$Q' = \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix} = -Q$$

$\therefore\ Q = \frac{1}{2}(A - A')$ is skew symmetric matrix.

$$\text{Now } P + Q = \begin{bmatrix} 1 & \frac{11}{2} \\ \frac{11}{2} & 7 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} = A$$

35. Differentiate $(\log x)^{\cos x}, x > 0$ **with respect to x.**

Solution:

$y = (\log x)^{\cos x}$

Applying log on both sides,

$\log y = \log(\log x)^{\cos x}$

$\log y = (\cos x) \times (\log(\log x))$

Differentiating on both sides with respect to x

$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \cos x \times \frac{1}{\log x} \times \frac{1}{x} + \log(\log x) \times (-\sin x)$$

$$\frac{dy}{dx} = y\left[\frac{\cos x}{x \log x} - \sin x(\log(\log x))\right]$$

$$\frac{dy}{dx} = (\log x)^{\cos x}\left[\frac{\cos x}{x \log x} - \sin x(\log(\log x))\right]$$

36. Find $\frac{dy}{dx}$ **if** $x = a(\theta + \sin\theta), y = a(1 - \cos\theta)$

Solution:

$x = a(\theta + \sin\theta)$

D.w.r. to θ

$\frac{dx}{d\theta} = a(1 + \cos\theta)$

$y = a(1 - \cos\theta)$

D.w.r. to θ

$\frac{dy}{d\theta} = a\sin\theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$$

$$\frac{dy}{dx} = \tan\frac{\theta}{2}$$

37. Find two positive numbers x & y such that x +y = 60 and xy^3 is maximum.

Solution :

$\text{Let } P = xy^3 \qquad x + y = 60 \ (\text{given})$

$= (60 - y)y^3 \qquad (\because y = 60 - x)$

$P = 60y^3 - y^4$

Differentiate w. r. to y

$\frac{dP}{dy} = 60 \times 3y^2 - 4y^3$

$= 180y^2 - 4y^3$

$\frac{d^2P}{dy^2} = 360y - 12y^2 = y(360 - 12y)$

For the value to be max/ min $\frac{dP}{dy} = 0$

$\frac{dP}{dy} = 0 \quad \Rightarrow \quad 180y^2 - 4y^3 = 0$

$\Rightarrow \quad 180y^2 = 4y^3$

$180 = 4y$

$y = \frac{180}{4} = 45$

$x = 60 - y = 60 - 45 = 15$

$\therefore \quad P$ is maximum

when $x = 45, y = 15$ or $x = 15, y = 45.$

38. Find $\int x \tan^{-1} x \, dx.$

Solution :

$\int x \tan^{-1} x \, dx.$

By using integration by parts

$I = \tan^{-1}x . \frac{x^2}{2} - \int \frac{x^2}{2} . \frac{1}{1+x^2} dx$

$= \frac{x^2}{2} \tan^{-1}x - \frac{1}{2} \int \frac{x^2 + 1 - 1}{x^2 + 1} dx$

$= \frac{x^2}{2} \tan^{-1}x - \frac{1}{2} \left[\int \left(1 - \frac{1}{x^2 + 1} \right) \right] dx$

$= \frac{x^2}{2} \tan^{-1}x - \frac{x}{2} + \frac{\tan^{-1}x}{2} + C$

39. Find the equation of a curve passing through the point (0, 1) and whose differential equation is given by $\frac{dy}{dx} = y \tan x \left(y \neq 0 \ and \ 0 \leq x < \frac{\pi}{2} \right)$

Solution:

$$\frac{dy}{dx} = y\tan x$$
$$\frac{dy}{y} = \tan x dx$$
$$\int \frac{dy}{y} = \int \tan x dx$$
$$\log y = \log|\sec x| + C$$
$\because (0,1)$ *passes through the curve*
$$\log 1 = \log|\sec 0| + c$$
$$0 = \log(1) + c$$
$$\Rightarrow c = 0$$
$$\therefore \log y = \log|\sec x|$$
$$y = \sec x$$

40. If $\vec{a}, \vec{b}$ *and* $\vec{c}$ be three vectors such that $|\vec{a}| = 3, |\vec{b}| = 4, |\vec{c}| = 5$ *and* $\vec{a}$ is perpendicular to $(\vec{b}+\vec{c})$, $\vec{b}$ is perpendicular to $(\vec{c}+\vec{a})$ *and* $\vec{c}$ is perpendicular to $(\vec{a}+\vec{b})$ then find $|\vec{a}+\vec{b}+\vec{c}|$.

Solution:

Given

$$\vec{a}.(\vec{b}+\vec{c}) = 0,\ \vec{b}.(\vec{c}+\vec{a}) = 0,\ \vec{c}.(\vec{a}+\vec{b}) = 0$$

$$|\vec{a}+\vec{b}+\vec{c}|^2 = (\vec{a}+\vec{b}+\vec{c})^2 = (\vec{a}+\vec{b}+\vec{c}).(\vec{a}+\vec{b}+\vec{c})$$
$$= \vec{a}.\vec{a} + \vec{a}.(\vec{b}+\vec{c}) + \vec{b}.\vec{b} + \vec{b}.(\vec{a}+\vec{c}) + \vec{c}.(\vec{a}+\vec{b}) + \vec{c}.\vec{c}$$
$$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 = 9 + 16 + 25 = 50$$

Therefore , $|\vec{a}+\vec{b}+\vec{c}| = \sqrt{50} = 5\sqrt{2}$

41. Find the area of the triangle having the points A(1, 1, 2), B(2, 3, 5) and C(1, 5, 5).

Solution:

We have

$\overrightarrow{OA} = \hat{i} + \hat{j} + 2\hat{k}$ $\quad \overrightarrow{OB} = 2\hat{i} + 3\hat{j} + 5\hat{k}$ $\quad$ and $\quad \overrightarrow{OC} = \hat{i} + 5\hat{j} + 5\hat{k}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = 4\hat{j} + 3\hat{k}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 0 & 4 & 3 \end{vmatrix}$$

$$= i(6 - 12) - j\,(3) + k\,(4)$$

$$= -6\hat{i} - 3\hat{j} + 4\hat{k}$$

$$|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{(-6)^2 + (-3)^2 + 4^2} = \sqrt{36+9+16} = \sqrt{61}$$

$$\therefore \quad \text{Area of triangle} = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{\sqrt{61}}{2} \text{ Sq.units.}$$

42. In answering a question on a multiple choice test, a student either knows the answer or guesses. Let $\frac{3}{4}$ be the probability that he knows the answer and $\frac{1}{4}$ be the probability that he guesses. Assuming that a student who guesses at the answer will be correct with probability $\frac{1}{4}$. What is the probability that the student knows the answer given that he answered it correctly?

Solution:

Let E_1 : the event that student knows the answer $\Rightarrow P(E_1) = \frac{3}{4}$

E_2 : Event that student guesses the answer

$\Rightarrow P(E_2) = \frac{1}{4}$

A: Event that the answer is correct.

Clearly, E_1 and E_2 are mutually exclusive and exhaustive

$P(A|E_1) = P$ (Answer is correct given that the student has known the answer) = 1 ($\because$ answer is correct is a sure events when the student knows the answer)

$P(A|E_2) = P$ (Event that the answer is correct given that the student guessed the answer)= $\frac{1}{4}$

Now, $P(E_1|A) = P($ the event that student knows the answer given that the answer is correct).

By Baye's theorem,

$$P(E_1|A) = \frac{P(E_1).P(A|E_1)}{P(E_1).P(A|E_1) + P(E_2).P(A|E_2)}$$

$$= \frac{\left(\frac{3}{4}\right).1}{\left(\frac{3}{4}\right).1 + \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)} = \frac{12}{12+1} = \frac{12}{13}$$

PART – D

Answer any four questions **(4 × 5 = 20)**

43. **Let $f : N \to Y$ be a function defined as $f(x) = 4x + 3$, where $Y = \{y : y = 4x + 3, x \in N\}$. Show that f is invertible. Find the inverse of f.**

Solution:

Consider an arbitrary elements y in Y.

Given function is f(x) = 4x+3 for some x in the domain N. $\Rightarrow y = 4x + 3$

$\Rightarrow \quad 4x = y - 3, \Rightarrow x = \frac{y-3}{4} \ \forall y \in Y$

Let us define the function

$g : Y \to N$, defined by $g(y) = \frac{y-3}{4}$.

Now $fog(y) = f(g(y)) = f\left(\frac{y-3}{4}\right)$

$= 4\left(\frac{y-3}{4}\right) + 3 = y$

$\therefore \quad fog(y) = I_Y$

And $gof(x) = g(4x+3) = \frac{4x+3-3}{4} = \frac{4x}{4} = x$

$\therefore \quad gof(x) = I_x$

This shows that $fog(y) = I_Y$ and $gof(x) = I_N$

Hence the given function is invertible with $f^{-1} = g$.

Hence the given function is invertible.

Let $f(x) = y \quad \Rightarrow \quad f^{-1}(y) = x.$

$\Rightarrow \quad f^{-1}(y) = \frac{y-3}{4} \quad \because \quad f^{-1}(x) = \frac{x-3}{4}$

44. **If** $A = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}$ **,** $C = \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$, **then compute** $(\mathbf{A+B})$ **and** $(\mathbf{B-C})$. **Also verify that A+(B - C) = (A+B)-C.**

Solution:

$$A + B = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 1 & -1 \\ 9 & 2 & 7 \\ 3 & -1 & 4 \end{bmatrix}$$

$$B - C = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -2 & 0 \\ 4 & -1 & 3 \\ 1 & 2 & 0 \end{bmatrix}$$

$$A + (B-C) = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} -1 & -2 & 0 \\ 4 & -1 & 3 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -3 \\ 9 & -1 & 5 \\ 2 & 1 & 1 \end{bmatrix}$$

$$(A+B) - C = \begin{bmatrix} 4 & 1 & -1 \\ 9 & 2 & 7 \\ 3 & -1 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -3 \\ 9 & -1 & 5 \\ 2 & 1 & 1 \end{bmatrix}$$

$\therefore \quad$ A+ (B - C) = (A + B) -C.

45. **Solve the following system of linear equations by matrix method**

$\mathbf{x+y+z=6, \quad y+3z=11 \quad and \quad x-2y+z=0}$

Solution:

Let first, second and third numbers be denoted by x, y and z respectively then according to given condition, we have

$x + y + z = 6$

$y + 3z = 11$

$x + z = 2y \Rightarrow x - 2y + z = 0$

Thus, system can be written in the form of AX = B

Where $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$

$$A=\begin{vmatrix}1 & 1 & 1\\ 0 & 1 & 3\\ 1 & -2 & 1\end{vmatrix}$$

$$|A| = 1[1+6]-1[0-3]+1[0-1]$$

$$|A| = 9 \neq 0$$

Hence, A is non-singular and so its inverse exists.

Now,

$$\text{co-factor of } A \quad \begin{matrix} A_{11}=7 & A_{12}=3 & A_{13}=-1\\ A_{21}=-3 & A_{22}=0 & A_{23}=3\\ A_{31}=2 & A_{32}=-3 & A_{33}=1\end{matrix}$$

$$\text{co-factor matrix } A=\begin{bmatrix}7 & 3 & -1\\ -3 & 0 & 3\\ 2 & -3 & 1\end{bmatrix}$$

$$\therefore \quad \text{adj}A=\begin{bmatrix}7 & -3 & 2\\ 3 & 0 & -3\\ -1 & 3 & 1\end{bmatrix}$$

$$\therefore \quad A^{-1}=\frac{\text{adj}A}{|A|}=\frac{1}{9}\begin{bmatrix}7 & -3 & 2\\ 3 & 0 & -3\\ -1 & 3 & 1\end{bmatrix}$$

$$\therefore \quad X=A^{-1}B \Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{9}\begin{bmatrix}7 & -3 & 2\\ 3 & 0 & -3\\ -1 & 3 & 1\end{bmatrix}\begin{bmatrix}6\\ 11\\ 0\end{bmatrix}$$

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{9}\begin{bmatrix}9\\ 18\\ 27\end{bmatrix}=\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$$

$$x=1 \qquad y=2 \qquad z=3$$

46. **If $y = Ae^{mx} + Be^{nx}$, show that $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$.**

Solution:

$y = Ae^{mx} + Be^{nx}$

Differentiate.w.r.to x

$$\frac{dy}{dx} = Ae^{mx}m + Be^{nx}n$$

Again differentiate w.r.t.x

$$\frac{d^2y}{dx^2} = Am^2e^{mx} + Bn^2e^{nx}$$

$$\text{LHS} = \frac{d^2y}{dx^2} - (m+n) + \frac{dy}{dx} + mny$$

$$= Am^2e^{mx} + Bn^2e^{nx} - (m+n)\left(Ame^{mx} + Bne^{nx}\right) + mn\left(Ae^{mx} + Be^{nx}\right)$$

$$= Am^2e^{mx} + Bn^2e^{nx} - Am^2e^{mx} - Bmne^{nx} - Amne^{mx} - Bn^2e^{nx} + Amne^{mx} + Bmne^{nx} = 0$$

47. **Find the integral value of $\frac{1}{x^2+a^2}$ with respect to x and hence find $\int \frac{dx}{x^2+2x+2}$**

Solution:

$$\text{Let } I = \int \frac{dx}{a^2+x^2}$$

Put $x = a\tan\theta$

$\theta = \tan^{-1}\frac{x}{a}$

D.w.r to θ

$dx = a\sec^2\theta \; d\theta$

$$= \int \frac{a\sec^2\theta \; d\theta}{a^2+a^2\tan^2\theta}$$

$$= \int \frac{a\sec^2\theta \; d\theta}{a^2\left(1+\tan^2\theta\right)}$$

$$= \frac{1}{a}\int d\theta$$

$$= \frac{1}{a}\theta + c$$

$$= \frac{1}{a}\tan^{-1}\frac{x}{a} + c$$

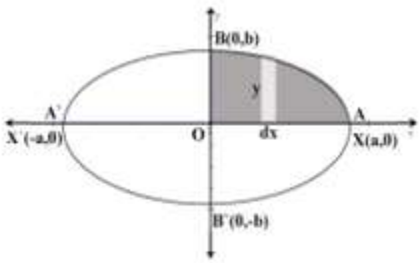
$$\int \frac{dx}{x^2+2x+2} = \int \frac{dx}{(x+1)^2+1}$$

$$= \frac{1}{1}\tan^{-1}\left(\frac{x+1}{1}\right) + c$$

$$= \tan^{-1}(x+1) + c$$

48. **Find the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ by method of integration**

Solution:



Area of ellipse = 4 (Area of the region ABOA).

$$\text{Area of region ABOA} = \int_0^a y \, dx$$

$$\text{Now, } \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\frac{y^2}{b^2} = \frac{a^2-x^2}{a^2}$$

$$y^2 = \frac{b^2}{a^2}\left(a^2-x^2\right)$$

$$y = \frac{b}{a}\sqrt{a^2-x^2}$$

$$\text{Area of region ABOA} = \frac{b}{a}\int_0^a \sqrt{a^2-x^2}\, dx$$

$$=\frac{b}{a}\left(\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right)_0^a$$

$$=\frac{b}{a}\left[\left(\frac{a}{2}\times 0+\frac{a^2}{2}\sin^{-1}1\right)-0\right]$$

$$=\frac{b}{a}\left(\frac{\pi a^2}{4}\right)$$

$$\therefore \quad \text{Area of ellipse} = 4\left(\frac{b}{a}\right)\frac{\pi a^2}{4}=\pi ab$$

49. **Find the particulars solution of the differential equation $\left(1+x^2\right)\frac{dy}{dx}+2xy=\frac{1}{1+x^2}$:y = 0 when $x=1$**

Solution:

The given Differential equation.

$$\left(1+x^2\right)\frac{dy}{dx}+2xy=\frac{1}{1+x^2}$$

Divide by $\left(1+x^2\right)$,

$$\frac{dy}{dx}+\frac{2x}{1+x^2}y=\frac{1}{\left(1+x^2\right)^2}$$

Now it is of the form

$$\frac{dy}{dx}+Py=Q \text{ where } P=\frac{2x}{1+x^2} \;\&\; Q=\frac{1}{\left(1+x^2\right)^2}$$

Thus it is a line as D.E in y

$$\therefore \quad I.F=e^{\int pdx}=e^{\int\frac{2x}{1+x^2}dx}=e^{\log\left(1+x^2\right)}=\left(1+x^2\right)$$

$\therefore$ Solution of the D.E is

$$y(I.F)=\int Q(I.F)dx+c$$

$$y\left(1+x^2\right)=\int\frac{1}{\left(1+x^2\right)^2}\left(1+x^2\right)dx+c$$

$$y\left(1+x^2\right)=\int\frac{1}{\left(1+x^2\right)}dx+c$$

$$y\left(1+x^2\right)=\tan^{-1}x+c \text{ --------(1)}$$

When y= 0, x=1

$$\Rightarrow \qquad 0=\tan^{-1}1+c \quad \therefore \quad c=\frac{-\pi}{4}$$

Substituting C value is (1) we get

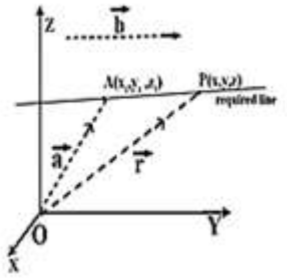
$$y\left(1+x^2\right)=\tan^{-1}x-\frac{\pi}{4}$$

Which is the particulars solution of the given
Differential equation.

50. **Derive the equation of line in space passing through a point and parallel to the vector both in vector and Cartesian form.**

Solution:

Let $\vec{a}$ be the position vector of the given point A with respect to the origin O. let 'l' be the line passes through the point A and is parallel to a given vector $\vec{b}$. Let $\vec{r}$ be the position vectors of any point P on the line.



Then $\overrightarrow{AP}$ is parallel to $\vec{b}$,

We have $\overrightarrow{AP} = \lambda\vec{b}$

$\Rightarrow \overrightarrow{OP} - \overrightarrow{OA} = \lambda\vec{b}$

$\Rightarrow \vec{r} - \vec{a} = \lambda\vec{b}$

$\Rightarrow \vec{r} = \vec{a} + \lambda\vec{b}$

This gives the position vector of any point P on the line.
Hence it is called vector equation of the line.
Cartesian form:
Let the coordinates of the given points
$A(x_1, y_1, z_1)$ and the direction ratios of the line be a , b, c. consider the coordinates of any point $P(x, y, z)$.

Then $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and

$\vec{b} = a\hat{i} + b\hat{j} + c\hat{k}$

Substituting in $\vec{r} = \vec{a} + \lambda\vec{b}$, we get

$x\hat{i} + y\hat{j} + z\hat{k} = \left(x_1\hat{i} + y_1\hat{j} + z_1\hat{k}\right) + \lambda\left(a\hat{i} + b\hat{j} + c\hat{k}\right)$

$x\hat{i} + y\hat{j} + z\hat{k} = (x_1 + a\lambda)\hat{i} + (y_1 + b\lambda)\hat{j} + (z_1 + c\lambda)\hat{k}$ equating their components.

We get $x = x_1 + a\lambda,\ y = y_1 + b\lambda,\ z = z_1 + c\lambda \Rightarrow \quad \lambda = \frac{x - x_1}{a},\quad \lambda = \frac{y - y_1}{b},\quad \lambda = \frac{z - z_1}{c}$

$\Rightarrow \frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$

This is the Cartesian equation of the line.

PART – E

Answer the following questions:

51. **a) Prove that** $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ and **hence evaluate** $\int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x.\cos x} dx$ **(6)**

Solution:

Consider $\int_0^a f(a-x)dx$

When $x = a,\ t = 0,$
and $x = 0,\ t = a$

Put $a - x = t$ in RHS
$dx = -dt$

$$= -\int_a^0 f(t)(dt)$$

$$= \int_0^a f(t)dt \quad \left[\because \int_0^a f(x\,dx) = -\int_a^0 f(x)dx\right] \qquad \int_0^a f(t)dx = \int_0^a f(x)\,dx$$

$$\therefore \quad \int_0^a f(x)dx = \int_0^a f(a-x)dx$$

$$\text{Let } I = \int_0^{\pi/2} \frac{\sin x - \cos x}{1+\sin x.\cos x}dx$$

$$I = \int_0^{\pi/2} \frac{\sin\left(\pi/2 - x\right) - \cos\left(\pi/2 - x\right)}{1+\sin\left(\pi/2 - x\right).\cos\left(\pi/2 - x\right)}dx$$

$$I = \int \frac{\cos x - \sin x}{1+\cos x.\sin x}dx \qquad \rightarrow (2)$$

add $(1) \& (2)$

$$2I = \int_0^{\pi/2} \frac{\sin x - \cos x + \cos x - \sin x}{1+\sin x.\cos x}dx$$

$$2I = \int_0^{\pi/2} 0.dx \Rightarrow I = 0$$

OR

b) Solve the following linear Programming Problem graphically:

Maximize Z = 4x + y ------ **(1)**

Subject to the constraints

$\mathbf{x + y \le 50;}$ ---- (2)

$\mathbf{3x + y \le 90}$ **---(3)**

$\mathbf{x \ge 0; y \ge 0}$ **---(4)**

Solution:

We have to minimize Z = 4x +y

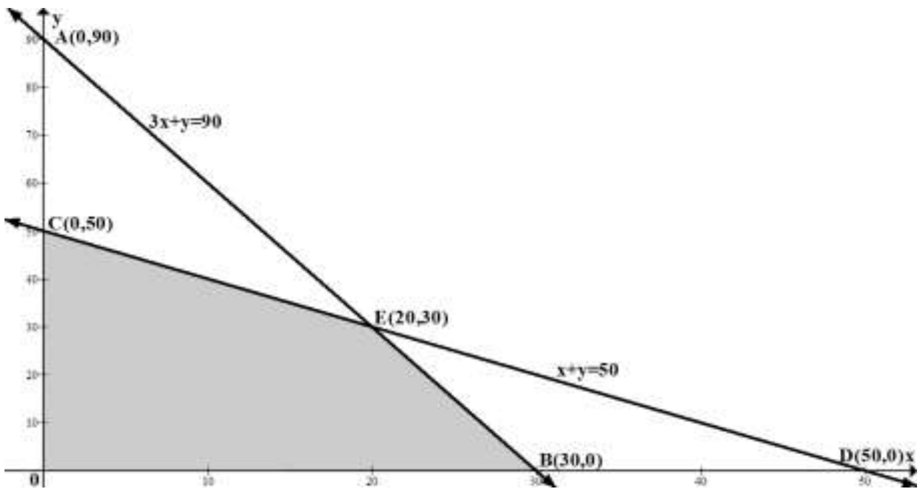
Now changing the given in equation $x + y \le 50$ ---------- (1)

$3x + y \le 90$ ------- (2) $x, y \ge 0$ --------- (3)

To equation,

$3x+y=90$		
x	0	30
y	90	0

$x+y=50$		
x	0	50
y	50	0



The shaded region in the above fig is feasible region determined by the system of constraints (1) to (3).

It is observed that the feasible region is bounded. The coordinates of the corner point OBEC are (0, 0), (30, 0) (20 , 30) and (0, 50)

The maximum value of Z = 4x +y

Corner point	Z = 4x + y
(0 , 0)	Z = 0
(30 , 0)	Z = 120 **maximum**
(20 ,30)	Z = 110
(0 , 50)	Z = 50

$\therefore Z_{\max} = 120$ at the point $(30, 0)$

52. **a) Show that the matrix $A=\begin{bmatrix}3 & 1\\-1 & 2\end{bmatrix}$ satisfies the equation $A^2 - 5A + 7I = 0$, where I is 2×2 identity matrix and O is 2×2 zero matrix. Using this equation find A^{-1} . (4)**

Solution:

We have, $\begin{bmatrix}3 & 1\\-1 & 2\end{bmatrix}$

$\therefore A^2 = AA = \begin{bmatrix}3 & 1\\-1 & 2\end{bmatrix}\begin{bmatrix}3 & 1\\-1 & 2\end{bmatrix} = \begin{bmatrix}8 & 5\\-5 & 3\end{bmatrix}$

$So, A^2 - 5A + 7I = \begin{bmatrix}8 & 5\\-5 & 3\end{bmatrix} - 5\begin{bmatrix}3 & 1\\-1 & 2\end{bmatrix} + 7\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}$

$= \begin{bmatrix}8-15+7 & 5-5+0\\-5+5+0 & 3-10+7\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = 0$

Now, $A^2 - 5A + 7I = 0$

$\Rightarrow A^{-1}\left(A^2 - 5A + 7I\right) = A^{-1}(0)$

[Multiplying throughout by A^{-1}]

$\Rightarrow A^{-1}A^2 - 5A^{-1}A + 7A^{-1}I = 0$

$\Rightarrow A - 5I + 7A^{-1} = 0$

$\Rightarrow 7A^{-1} = 5I - A$

$7A^{-1} = \begin{bmatrix}5 & 0\\0 & 5\end{bmatrix} - \begin{bmatrix}3 & 1\\-1 & 2\end{bmatrix} = \begin{bmatrix}2 & -1\\1 & 3\end{bmatrix}$

$\Rightarrow A^{-1} = \frac{1}{7}\begin{bmatrix}2 & -1\\1 & 3\end{bmatrix}$

OR

b) **Find the value of K so that the function f defined as $f(x) = \begin{cases} Kx+1, & \text{if } x \le \pi \\ \cos x, & \text{if } x > \pi \end{cases}$ is continuous at $x = \pi$.**

Solution:

The function is defined at $x = \pi$.

i.e., $f(x) = Kx + 1 \Rightarrow f(\pi) = K\pi + 1$

$LHL = \lim_{x\to\pi^-} f(x) = \lim_{x\to\pi+} Kx + 1 = K\pi + 1$

$$R\,H\,L = \lim_{x \to \pi^-} Kx + 1 = \lim_{x \to \pi +} \cos x = \cos\pi$$

f is continuous at $x = \pi$

$\therefore \quad K\pi + 1 = \cos\pi$

$$K\pi + 1 = -1$$

$$K = \frac{-2}{\pi}$$

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